LOCALIZATION FOR FOURIER SERIES ON $SU(2)$

BY

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1. **Introduction.** Let $SU(2)$ be the group of 2×2 unitary matrices with determinant 1, and for each positive integer n let χ_n be the irreducible n-dimensional character of $SU(2)$, ([6, pp. 151, 163] our character χ_n is Weyl's χ_{n-1}). With each integrable function f on $SU(2)$ there is associated a Fourier series

$$f \sim \sum_{n=1}^{\infty} P_n f, \qquad P_n f = f * n\chi_n,$$

where $*$ denotes convolution. For each $b \in SU(2)$ and each integer $k>0$ we set

$$S_k(f:b) = \sum_{n=1}^{k} P_n f(b).$$

Let $N(b)$ be the space of functions $f \in L^1(SU(2))$ which vanish on a neighborhood V_f of b. The main results of this paper are Theorems A–C below.

THEOREM A. *Let $b \in SU(2)$, $f \in N(b)$. If all of the first derivatives of f (in the distribution sense) are functions in $L^p(SU(2))$ for some $p>3/2$, then* $\lim_{n\to\infty} S_n(f:b)=0$. *If $p<3/2$, there is a function $f\in N(b)$ whose first derivatives are all functions in $L^p(SU(2))$ and such that* $\lim_{n\to\infty} S_n(f:b)$ *does not exist.*

THEOREM B. *If $f \in N(b) \cap N(-b)$ and the first derivatives of f are functions in $L^1(SU(2))$, then* $\lim_{n\to\infty} S_n(f:b)=0$.

Say a function $f \in L^1(SU(2))$ is of bounded variation if its first derivatives are all measures.

THEOREM C. *If $b \in SU(2)$ and $V \subset SU(2)$ is any nonvoid open set, then there is a function of bounded variation which vanishes on the complement of V such that* $\lim_{n\to\infty} S_n(f:b)$ *does not exist.*

The proof of Theorem C yields the following example. Let e be the identity for $SU(2)$ and let

$$\begin{aligned} g(b) &= 0 && \text{if } \operatorname{tr}(b) < 0,\\ &= \tfrac{1}{2} && \text{if } \operatorname{tr}(b) = 0,\\ &= 1 && \text{if } \operatorname{tr}(b) > 0, \end{aligned}$$

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where tr = trace. Then $\lim_{n\to\infty} S_n(g:b)=g(b)$ if $b\neq \pm e$, and $\lim_{n\to\infty} S_n(g:b)$ does not exist if $b=\pm e$. (Topologically $SU(2)$ is a 3-sphere, $g=1$ on the northern hemisphere, $\frac{1}{2}$ on the equator, and 0 on the southern hemisphere, and the Fourier series for f diverges at the north and south poles.)

2. **Influence sets.** Let G be a compact group, let $S=\{S_\alpha\}$ $(\alpha\in A)$ be a summation method for G, [3, §5], let M be a linear submanifold of $L^1(G)$, let F be a closed subset of G and let $b\in G$. We will say F influences S at b for functions in M if $\lim_{\alpha\in A} S_\alpha(f:b)=0$ for every $f\in M$ which vanishes on a neighborhood of F.

LEMMA 2.1. *Let G be a compact Lie group, and let M be a left translation invariant linear submanifold of $L^1(G)$ which is also a module over $C^\infty(G)$ (i.e., $fg\in M$ for all $f\in M$, $g\in C^\infty(G)$). Let S be a summation method for G, $b\in G$. Then there exists a unique closed set $I_b(S, M)$ in G such that $I_b(S, M)$ influences S at b for functions in M and $I_b(S, M)$ is contained in every closed set which influences S at b for functions in M. Also $I_b(S, M)=bI_e(S, M)$ where e is the identity for G.*

Proof. This was proved in [3, Theorem 5.7] for the case where $M=L^1(G)$. We will show that if F_1 and F_2 influence S at b for functions in M, then so does $F_1\cap F_2$. The rest of the proof is exactly like the proof given in [3]. Let F_1, F_2 influence S at b for functions in M and let f be a function in M which vanishes on a neighborhood U of $F_1\cap F_2$. Let V_1, V_2 be disjoint compact neighborhoods of F_1-U, F_2-U respectively and let h be a C^∞ function on G such that $h=1$ on V_1 and $h=0$ on V_2. Then $fh\in M$ and fh vanishes near F_2 so $\lim_{\alpha\in A} S_\alpha(fh:b)=0$. Similarly $\lim_{\alpha\in A} S_\alpha(f(1-h):b)=0$, so $\lim_{\alpha\in A} S_\alpha(f:b)=0$ and $F_1\cap F_2$ influences b for functions in M.

We will write $I_e(S, M)=I(S, M)$, and we say that the method S has the localization property for functions in M if $I(S, M)=\{e\}$. An element b of G is in $I(S, M)$ if and only if for each neighborhood V of b there is a function $f\in M$ such that f vanishes on $G-V$ and $\lim_{\alpha\in A} S_\alpha(f:e)\neq 0$.

LEMMA 2.2. *Let G be a compact Lie group, let $S=\{S_\alpha\}$, $(\alpha\in A)$ be a central summation method for G* [3, §5] *and let M be a linear submanifold of $L_1(G)$ which is left and right translation invariant and which is a module over $C^\infty(G)$. Then $aI(S, M)a^{-1}=I(S, M)$ for all $a\in G$.*

Proof. Let $b\in I(S, M)$, $a\in G$ and let V be any neighborhood of aba^{-1}. Then $a^{-1}Va$ is a neighborhood of b, so there is a function $f\in M$ such that f vanishes on $G-a^{-1}Va$ and $\lim_{\alpha\in A} S_\alpha(f:e)\neq 0$. Define $g\in M$ by $g(x)=f(a^{-1}xa)$ for all $x\in G$. Then g vanishes on $G-V$ and by [3, Equation (5.2)]

$$S_\alpha(g:e) = g * S_\alpha^\vee(e) = \int_G f(a^{-1}ya)S_\alpha^\vee(y^{-1})\,dy = \int_G f(y)S_\alpha^\vee(ay^{-1}a^{-1})\,dy$$

$$= \int_G f(y)S_\alpha^\vee(y^{-1})\,dy = f * S_\alpha^\vee(e) = S_\alpha(f:e).$$

Thus $\lim_{\alpha\in A} S_\alpha(g:e)\neq 0$ and $aba^{-1}\in I(S, M)$.

Let G be a compact Lie group and let $\mathfrak{G}$ be the Lie algebra of G, i.e., the Lie algebra of left invariant vector fields on G. A function $f\in L^1(G)$ is of *bounded variation* if for each $D\in\mathfrak{G}$ there is a constant K_D such that

$$(2.3)\qquad |(Dg,f)| \leqq K_D\|g\|_\infty \qquad \text{for all } g\in C^\infty(G).$$

The space of functions of bounded variation on G will be denoted by BV. For $1\leqq p\leqq\infty$ let W_p^1 be the space of functions in $L^1(G)$ all of whose first derivatives are functions in $L^p(G)$. (A function $f\in L^1(G)$ is in W_p^1 if and only if for each $D\in\mathfrak{G}$ there is a function $Df\in L^p(G)$ such that $(Dg,f)=-(g,Df)$, for all $g\in C^\infty(G)$.) It is routine to verify that BV and W_p^1 are left and right translation invariant modules over $C^\infty(G)$.

3. **Localization theorems.** In this section G will always denote $SU(2)$ and $\mathfrak{G}$ will be the Lie algebra of $SU(2)$. For each integer $n\geqq 1$ let E_n be the n^2 dimensional two-sided ideal in $L^2(G)$, let χ_n be the irreducible character in E_n and let P_n be the orthogonal projection onto E_n. Let $S=\{S_n\}$ $(1\leqq n<\infty)$, be the summation method for G defined by

$$(3.1)\qquad S_n = P_1+\cdots+P_n.$$

In [3] it was shown that $I(S,L^1(G))=G$. In this section we find $I(S, W_p^1)$ and $I(S,BV)$.

Theorem 3.2. *Let S be the summation method defined in* (3.1). *Then*

$$I(S\colon W_p^1) = \{e\} \qquad \text{if } p>3/2,$$
$$= \{e\}\cup\{-e\} \qquad \text{if } 1\leqq p<3/2.$$

The proof will require a few lemmas.

Lemma 3.3. *For any $D\in\mathfrak{G}$*

$$(3.4)\qquad (D\chi_2)n\chi_n = D(\chi_{n+1}-\chi_{n-1}) \qquad \textit{for all } n\geqq 2.$$

$$(3.5)\qquad (D\chi_2)\sum_{k=1}^{n} k\chi_k = D(\chi_n+\chi_{n+1}) \qquad \textit{for all } n\geqq 1.$$

$$(3.6)\qquad (D\chi_n)(3-\chi_3) = ((n+1)\chi_{n-1}-(n-1)\chi_{n+1})\cdot D\chi_2 \qquad \textit{for all } n\geqq 2.$$

Proof. We prove (3.4) by induction on n. Observe that (3.4) holds for $n=1$ if we set $\chi_0=0$. We will use the relations

$$(3.7a)\qquad \chi_2\chi_{k+1} = \chi_k+\chi_{k+2},$$

$$(3.7b)\qquad \chi_3\chi_k = \chi_{k+2}+\chi_k+\chi_{k-2}, \qquad (k\geqq 2)$$

[6, p. 128]. For $n=2$ we have

$$D\chi_2\cdot 2\chi_2 = D(\chi_2^2) = D(\chi_3+1) = D(\chi_3-1).$$

Now assume (3.4) for $n \leqq k$ where $k \geqq 2$. Applying the derivation D to both sides of (3.7a) and using (3.4) for $n=k$ we get

$$D\chi_2 \cdot \chi_{k+1} + \chi_2(D\chi_2 \cdot k\chi_k + D\chi_{k-1}) = D\chi_k + D\chi_{k+2}. \tag{3.8}$$

Since

$$\chi_2 \cdot D\chi_{k-1} = D(\chi_2 \cdot \chi_{k-1}) - D\chi_2 \cdot \chi_{k-1} = D(\chi_k + \chi_{k-2}) - D\chi_2 \cdot \chi_{k-1}$$

we obtain

$$(k+1)\chi_{k+1} \cdot D\chi_2 + (k-1)\chi_{k-1} \cdot D\chi_2 = D\chi_{k+2} - D\chi_{k-2}.$$

If we use the induction hypothesis to evaluate $(k-1)\chi_{k-1} \cdot D\chi_2$ we obtain (3.4) for $n=k+1$. (3.5) follows immediately from (3.4). Finally

$$\begin{aligned} ((n+1)\chi_{n-1} - (n-1)\chi_{n+1})D\chi_2 &= ((n-1)\chi_{n-1} - (n+1)\chi_{n+1} + 2(\chi_{n-1} + \chi_{n+1}))D\chi_2 \\ &= D(\chi_n - \chi_{n-2}) - D(\chi_{n+2} - \chi_n) + 2\chi_2\chi_n D\chi_2 \\ &= D(-\chi_{n+2} + 2\chi_n - \chi_{n-2}) + \chi_n D\chi_2^2 \\ &= D((3-\chi_3)\chi_n) - \chi_n D(3-\chi_3) \\ &= (3-\chi_3)D\chi_n. \end{aligned}$$

LEMMA 3.9. *Let $h \in W_p^1$ where $p > 3/2$. Then*

$$\lim_{n \to \infty} n \int_G h(a)\chi_n(a)\, da = 0.$$

Proof. Let Δ be the Laplace operator for G, $\Delta = D_1^2 + D_2^2 + D_3^2$ where D_1, D_2, D_3 is any basis for $\mathfrak{G}$ which is orthonormal with respect to the Killing form for $\mathfrak{G}$. Each character χ_n is an eigenvector for Δ [1, p. 426]. Say $\Delta\chi_n = \lambda_n\chi_n$. Then it is well known that

$$\lambda_n = \lambda_2(n^2-1)/3. \tag{3.10}$$

((3.10) can be verified by induction using the relation $\lambda_n = \Delta\chi_n(e)/n$ together with (3.7a) and the fact that $D\chi_n(e)=0$ for all n since χ_n has a maximum at e.) For $h \in W_p^1$ we have

$$\begin{aligned} \int_G h(a)\chi_n(a)\, da &= 3 \int_G h(a)\, \Delta\chi_n(a)\, da/\lambda_2(n^2-1) \\ &= -3 \sum_{i=1}^{3} (D_i h, D_i\chi_n)/\lambda_2(n^2-1), \end{aligned}$$

so the lemma will follow if we show

$$\lim_{n \to \infty} (g, D\chi_n)/n = 0, \tag{3.11}$$

for all $g \in L^p(G)$, $D \in \mathfrak{G}$. Condition (3.11) is satisfied for any g in the representative ring of G, and since the representative ring is dense in $L^p(G)$ for $1 \leqq p < \infty$ it follows from [3, Lemma 5.10] that (3.11) holds for all $g \in L^p(G)$ $(3/2 < p < \infty)$ if and only if

$$\{n^{-1}\|D\chi_n\|_q : 1 \leqq n < \infty\} \tag{3.12}$$

is bounded for each q, $1 < q < 3$.

Now $\Delta\chi_2^2=2\chi_2\,\Delta\chi_2+2\sum_{i=1}^{3}(D_i\chi_2)^2$ so using (3.10) we get

$$\sum_{i=1}^{3}(D_i\chi_2)^2 = (\chi_3-3)\lambda_2/3. \tag{3.13}$$

Since $D_i\chi_2$ is real it follows that

$$(3-\chi_3)^{-1/2}D\chi_2\in L^\infty(G), \qquad \text{for all } D\in\mathfrak{G}. \tag{3.14}$$

By (3.6) and (3.14) we see that there is a constant K_D such that

$$\begin{aligned}\|D\chi_n\|_q &= \|((n+1)\chi_{n-1}-(n-1)\chi_{n+1})D\chi_2\cdot(3-\chi_3)^{-1}\|_q\\ &\leqq K_D\|((n+1)\chi_{n-1}-(n-1)\chi_{n+1})(3-\chi_3)^{-1/2}\|_q\end{aligned}$$

and hence

$$n^{-1}\|D\chi_n\|_q \leqq K_D\|\chi_{n-1}-\chi_{n+1}+n^{-1}\chi_2\chi_n\|_\infty\|(3-\chi_3)^{-1/2}\|_q. \tag{3.15}$$

Using the explicit formulas for the characters and Haar measure on $SU(2)$ discussed in [6, pp. 151, 163] we get $\chi_{n+1}(a)-\chi_{n-1}(a)=\varepsilon^n+\varepsilon^{-n}$ where the eigenvalues of a are ε, ε^{-1}, so

$$\|\chi_{n-1}-\chi_{n+1}+n^{-1}\chi_2\chi_n\|_\infty \leqq 4,$$

and

$$\|(3-\chi_3)^{-1/2}\|_q^q = 2^{1-q}\cdot\frac{1}{\pi}\int_0^\pi \sin^{2-q}(t)\,dt.$$

(3.12) thus follows from (3.15), and the proof of Lemma 3.9 is complete. (Note that $W_p^1\supset W_\infty^1$ for $1\leqq p\leqq\infty$.)

Let $f\in W_p^1$ where $p>3/2$ and suppose f vanishes on a neighborhood V of e. By [3, Equation 5.12] we have

$$S_n(f:e) = f * S_n^\vee(e) = \int_G f(a)((n+1)\chi_n(a)-n\chi_{n+1}(a))(2-\chi_2(a))^{-1}\,da.$$

Since f vanishes near e and $(2-\chi_2)^{-1}$ is infinitely differentiable except at e it follows that $h=(2-\chi_2)^{-1}f$ is a function in W_p^1 which vanishes near e, and

$$S_n(f:e) = \int_G h(a)((n+1)\chi_n(a)-n\chi_{n+1}(a))\,da.$$

By Lemma 3.9 $\lim_{n\to\infty} S_n(f:e)=0$, hence $I(S, W_p^1)=\{e\}$ if $p>3/2$.

LEMMA 3.16. *Let f be a function in $L^2(SU(2))$ which is continuously differentiable except at a single point a, and let $D\in\mathfrak{G}$. Suppose that the pointwise derivative Df (which is defined except at a) is a function in L^1. Then the derivative of f considered as a distribution is the function Df.*

(The proof is a standard kind of argument, and is omitted.)

LEMMA 3.17. *Suppose $1\leqq p<3/2$. Then there exists a function $f\in W_p^1$ which vanishes on a neighborhood of e such that $\lim_{n\to\infty} S_n(f:e)$ does not exist.*

Proof. Let F be the function on $[0, \pi)$ defined by

$$F(t) = 0 \quad \text{if } 0 \leqq t \leqq \pi/2$$
$$= \csc t - 1 \quad \text{if } \pi/2 \leqq t < \pi. \tag{3.18}$$

Let θ be the function on G defined by

$$\theta = \arc\cos(\tfrac{1}{2}\chi_2). \tag{3.19}$$

Let $f = F \circ \theta$. Then $f \in L^2(G) \subset L^p(G)$, [6, p. 163], and for any $D \in \mathfrak{G}$ we have

$$Df(a) = 0 \quad \text{if } 0 \leqq \theta(a) \leqq \pi/2$$
$$= -\csc\theta(a)\cot\theta(a) D\theta(a) \quad \text{if } \pi/2 \leqq \theta(a) < \pi. \tag{3.20}$$

Now

$$D\theta = -\tfrac{1}{2}(1-(\tfrac{1}{2}\chi_2)^2)^{-1/2} D\chi_2 = -(3-\chi_3)^{-1/2} D\chi_2$$

is bounded by (3.14), so $Df \in L^1(G)$, and by Lemma 3.16 the distribution derivative of f coincides with the function Df. Since $Df \in L^p(G)$ for $1 \leqq p < 3/2$, we have $f \in W_p^1$ for $1 \leqq p < 3/2$.

$$P_k f = (f, \chi_k)\cdot\chi_k = 2\chi_k/\pi k, \qquad k \text{ odd}, k > 1.$$

Hence $P_k f(e)$ does not tend to 0 as k becomes large and hence $\lim_{n\to\infty} S_n(f:e) = \lim_{n\to\infty} \sum_1^n P_k f(e)$ does not exist.

LEMMA 3.21. *Let $f \in L^1(G)$ and suppose that f vanishes on a neighborhood V of $\{e\} \cup \{-e\}$. Then*

$$\lim_{n\to\infty} \int_G f(a)\chi_n(a)\,da = 0. \tag{3.22}$$

Proof. Let $L^p(G-V)$ be the subspace of $L^p(G)$ consisting of those functions in $L^p(G)$ which vanish on V. Since $\{\chi_n\}$ is an orthonormal set in $L^2(G)$, (3.22) holds for all $f \in L^2(G-V)$ and since $L^2(G-V)$ is dense in $L^1(G-V)$ it follows from [3, Lemma 5.10] that (3.22) holds for all $f \in L^1(G-V)$ if and only if the set of numbers $\{\sup\{|\chi_n(a)| : a \in G-V\} : 1 \leqq n < \infty\}$ is bounded. It is easy to verify that this is the case.

LEMMA 3.23. $I(S: W_1^1) = \{e\} \cup \{-e\}$.

Proof. Let f be a function in W_1^1 which vanishes on a neighborhood V of $\{e\} \cup \{-e\}$. Then by (3.5)

$$S_n(f:e) = \int_G f(a)\left(\sum_{k=1}^n k\chi_k(a)\right) da = \int_G (f(a)/D\chi_2(a))\cdot D(\chi_n + \chi_{n+1})(a)\,da, \tag{3.24}$$

for any $D \in \mathfrak{G}$, $D \neq 0$. Let D_1, D_2, D_3 be an orthonormal basis for $\mathfrak{G}$ with respect to the Killing form for $\mathfrak{G}$, and let $F_j = \{a \in G : D_j\chi_2(a) = 0\}$ $(1 \leqq j \leqq 3)$. It follows from (3.13) and the fact that $\chi_3(a) = 3$ if and only if $a = \pm e$ that $F_1 \cap F_2 \cap F_3$

$=\{e\} \cup \{-e\}$. Let U be a neighborhood of $\{e\} \cup \{-e\}$ such that $U^- \subset V$ and let $F_j^U = F_j \cap (G-U)$ for $1 \leqq j \leqq 3$. Then each F_j^U is compact and

$$\bigcap_{j=1}^{3} F_j^U = \varnothing.$$

Choose open sets W_j $(1 \leqq j \leqq 3)$ in G so that

$$W_j \supset F_j^U \quad \text{and} \quad \bigcap_{j=1}^{3} W_j^- = \varnothing.$$

Let $(W_j^-)'$ be the complement of W_j^- so that $\{(W_j^-)' : 1 \leqq j \leqq 3\}$ is an open cover for $SU(2)$. Let $\{g_1, g_2, g_3\}$ be a C^∞ partition of unity for G subordinate to this cover (so $g_j = 0$ on W_j^-) and let h be a C^∞ function such that $h=0$ on U and $h=1$ on $G-V$. Let $\lambda_j = g_j h$ for $1 \leqq j \leqq 3$. Then we have

$$f = \sum_{j=1}^{3} \lambda_j f.$$

Moreover each $\lambda_j / D_j \chi_2$ is a C^∞ function since λ_j vanishes on a neighborhood of the zeros of $D\chi_2$. By (3.24)

$$S_n(f:e) = \sum_{j=1}^{3} S_n(\lambda_j f:e) = \sum_{j=1}^{3} \int_G (\lambda_j f / D_j \chi_2) \cdot D_j(\chi_{n+1} + \chi_n)\, d\mu.$$

Each function $\lambda_j f / D_j \chi_2$ is in W_1^1, since W_1^1 is a module over $C^\infty(G)$. Hence

$$S_n(f:e) = -\sum_{j=1}^{3} \int_G D_j(\lambda_j f / D_j \chi_2) \cdot (\chi_{n+1} + \chi_n)\, d\mu$$

and it follows from Lemma 3.21 that $\lim_{n\to\infty} S_n(f:e) = 0$. Hence $I(S, W_1^1) \subset \{e\} \cup \{-e\}$. By Lemma 3.17 $I(S, W_1^1)$ contains $\{e\}$ as a proper subset so Lemma 3.23 follows.

We have already observed that $I(S, W_p^1) = \{e\}$ if $p > 3/2$. The rest of Theorem 3.2 is clear from Lemma 3.17 and Lemma 3.23. If $f \in L^1(G)$ and f vanishes near e then either $\lim_{n\to\infty} S_n(f:e) = 0$ or $\lim_{n\to\infty} S_n(f:e)$ does not exist [3, Theorem 7.12]. Hence Theorems A and B of the introduction follow from Theorem 3.2.

THEOREM 3.25. *$I(S, BV) = G$. (This is Theorem C of the introduction.)*

Again the proof requires a few lemmas.

LEMMA 3.26. *Let f be a class function in $L^1(SU(2))$ and let f' be the even function on $[-\pi, \pi]$ defined by*

$$f'(t) = f(x_t) \qquad \textit{where } x_t = \operatorname{diag}(e^{it}, e^{-it}).$$

Suppose that $f' \in L^1(-\pi, \pi)$ and let $f' \sim \sum C_n e^{int}$ be the Fourier series for f'. Then

$$S_n(f:a) = \sum_{-n+1}^{n-1} C_k e^{ik\theta(a)} - (C_n \chi_{n-1}(a) + C_{n+1} \chi_n(a)) \tag{3.27}$$

for all $a \in SU(2)$, $n = 1, 2, \ldots$, (θ is as in (3.19)).

Proof. $C_n = C_{-n}$ so for any $n > 1$

$$\int_G f(a)\chi_n(a)\,da = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\sin nt \sin t\,dt = C_{n-1} - C_{n+1}. \tag{3.28}$$

Using the relation $\chi_n = 2\cos(n-1)\theta + \chi_{n-2}$, for $n \geqq 2$, ($\chi_0 = 0$) we get

$$P_n f = (f, \chi_n)\chi_n = 2C_{n-1}\cos(n-1)\theta + C_{n-1}\chi_{n-2} - C_{n+1}\chi_n \tag{3.29}$$

for $n \geqq 2$. Since $P_1 f = C_0 - C_2$ and

$$S_n(f:a) = \sum_{k=1}^{n} P_k f(a),$$

(3.27) follows from (3.29).

Lemma 3.30. *For each $t \in (-2, 2)$ let λ_t be the function on G defined by*

$$\begin{aligned} \lambda_t(a) &= 0 \quad \text{if } \chi_2(a) < t, \\ &= \tfrac{1}{2} \quad \text{if } \chi_2(a) = t, \\ &= 1 \quad \text{if } \chi_2(a) > t. \end{aligned} \tag{3.31}$$

Then the Fourier series for λ_t diverges at $\pm e$ and converges elsewhere.

Proof. λ_t' is clearly in $L^1(-\pi, \pi)$ and a straightforward calculation shows that

$$\lambda_t'(s) \sim \sum C_n e^{ins}, \tag{3.32}$$

where $C_n = \sin(n \arccos(\frac{1}{2}t))/\pi n$. We will write $T = \arccos(\frac{1}{2}t)$. By (3.27)

$$S_n(\lambda_t:a) = \sum_{-n+1}^{n-1} \frac{\sin kT}{\pi k} e^{ik\theta(a)} - \frac{\sin nT}{\pi n}\chi_{n-1}(a) - \frac{\sin(n+1)T}{\pi(n+1)}\chi_n(a).$$

We know that

$$\lim_{n\to\infty} \sum_{-n}^{n} \frac{\sin kT}{\pi k} e^{ik\theta(a)} = \lambda_t'(\theta(a))$$

for all $\theta(a)$, so $\lim_{n\to\infty} S_n(\lambda_t:a)$ exists if and only if the limit

$$\lim_{n\to\infty} (\sin nT)\chi_{n-1}(a)/n + (\sin(n+1)T)\chi_n(a)/(n+1) \tag{3.33}$$

exists. If $a \neq \pm e$ this limit is clearly 0. If $a = e$ (3.33) becomes

$$\lim_{n\to\infty} (n-1)(\sin nT)/n + n(\sin(n+1)T)/(n+1)$$

and this limit does not exist for $T \in (0, \pi)$. Similarly $\lim_{n\to\infty} S_n(\lambda_t:-e)$ does not exist.

In §4 we will show that the functions $\lambda_t \in BV$ by computing their derivatives explicitly. (The explicit formula for the derivatives will be used in a later paper.) For the present we assume $\lambda_t \in BV$.

Proof of Theorem 3.25. Every conjugacy class of $SU(2)$ is of the form

$$K_t = \{a \in SU(2) : \chi_2(a) = t\} \qquad \text{for some } t \in [-2, 2]. \tag{3.34}$$

Any neighborhood N of K_t contains a set of the form

$$N_{t\varepsilon} = \{a \in G : |\chi_2(a) - t| < \varepsilon\}.$$

Let $f_{t\varepsilon}$ be a continuously differentiable class function on G such that

$$\begin{aligned} f_{t\varepsilon}(a) &= 0 \qquad \text{if } \chi_2(a) \leqq t-\varepsilon, \\ &= -1 \qquad \text{if } \chi_2(a) \geqq t+\varepsilon, \end{aligned}$$

and such that $f_{t\varepsilon}=0$ near $-e$ and $f_{t\varepsilon}=-1$ near e. Then $f'_{t\varepsilon} \sim \sum C_n e^{ins}$ where $C_n = o(n^{-1})$ and it follows from (3.27) that $f_{t\varepsilon}$ has an everywhere convergent Fourier series. Let $t \in (-2, 2)$, let N be any neighborhood of K_t, and let $N_{t\varepsilon} \subset N$. Then $\lambda_t + f_{t\varepsilon} \in BV$, $\lambda_t + f_{t\varepsilon}$ vanishes on the complement of N and $\lim_{n\to\infty} S_n(\lambda_t + f_{t\varepsilon} : e) = \lim_{n\to\infty} S_n(\lambda_t : e)$ does not exist. Hence $N \cap I(S, BV) \neq \varnothing$ for every neighborhood N of K_t, hence $K_t \cap I(S, BV) \neq \varnothing$ and by Lemma 2.2, $K_t \subset I(S, BV)$. Clearly $K_t \subset I(S, BV)$ for $t = \pm 2$ and it follows that $I(S, BV) = G$.

4. **Calculation of some derivatives.** Let G be a compact Lie group with Lie algebra $\mathfrak{G}$. For each $D \in \mathfrak{G}$ let D_R be the right invariant vector field on G defined by $D_R = -JDJ$ where $(Jf)(x) = \bar{f}(x^{-1}) = f^*(x)$ for any function f on G.

LEMMA 4.1. *Let G be a compact connected semisimple Lie group with Lie algebra $\mathfrak{G}$, let $\mathfrak{A}$ be an abelian subalgebra of $\mathfrak{G}$, and let E be a minimal two-sided ideal in the convolution algebra $L^2(G)$, of dimension d_E^2. Then there exists an irreducible matrix representation $a \to (\mu_{kj}(a))$ of G and a family $\{\phi_k\}$ $(1 \leqq k \leqq d_E)$ of real linear functionals on $\mathfrak{G}$ such that*

$$\mu_{kj} \in E, \qquad 1 \leqq k, j \leqq d_E, \tag{4.2}$$

$$D\mu_{kj} = i\phi_j(D)\mu_{kj}, \qquad 1 \leqq k, j \leqq d_E, \quad D \in \mathfrak{A}, \tag{4.3}$$

$$D_R\mu_{kj} = i\phi_k(D)\mu_{kj}, \qquad 1 \leqq k, j \leqq d_E, \quad D \in \mathfrak{A}. \tag{4.4}$$

Proof. The space E is translation invariant, and hence is invariant under each operator $D \in \mathfrak{G}$. Also each restriction $D|_E$, $D \in \mathfrak{G}$ is skew Hermitian since $(Df, g) = -(f, Dg)$ for all $f, g \in E$. The complex associative algebra generated by $\{D|_E : D \in \mathfrak{A}\}$ is an abelian selfadjoint algebra of operators on E, and hence is generated by a single operator $T|_E$. Using left invariance of T it is easy to verify that $T(f * g) = f * Tg$ for all $f, g \in E$, and from this it follows that any eigenspace of $T|_E$ is a left ideal in E. Since E is the orthogonal sum of the eigenspaces of $T|_E$ and any left ideal in E is the orthogonal sum of minimal left ideals we can write $E = \sum E_k$ $(1 \leqq k \leqq d_E)$ where each E_i is a minimal left ideal invariant under T (and hence under $\mathfrak{A}$), and the ideals E_k are mutually orthogonal. Write $\mathfrak{A}_R = \{D_R : D \in \mathfrak{A}\}$. Each ideal E_k is invariant under $\mathfrak{A}_R$ ([4, p. 294] and note that $D_R P_E$ is a *bounded* right invariant operator where P_E is the projection onto E). Write $E_k = L^2(G) * f_k$ where $f_k = f_k^*$ is a minimal idempotent in $L^2(G)$, and let $F_k = J(E_k) = f_k * L^2(G)$. Then $F_k \subset E$ and each F_k is a minimal right ideal which is invariant under $\mathfrak{A}$ and $\mathfrak{A}_R$. $E_k \cap F_j = f_j * L^2(G) * f_k = f_j * E * f_k$ is a 1-dimensional subspace of E [2, p. 104], and the spaces $E_k \cap F_j$ are mutually orthogonal and invariant under $\mathfrak{A}$ and $\mathfrak{A}_R$. For each $D \in \mathfrak{A}$ and $1 \leqq k \leqq d_E$ let $\phi_k(D)$ be the number such that $D|_{E_k} = i\phi_k(D)I_k$, where I_k is the identity operator on E_k. $D|_{E_k}$ is a scalar by Schur's lemma and $\phi_k(D)$ is real because $D|_E$ is skew Hermitian. If $f \in F_k$ and $D \in \mathfrak{A}$ then $D_R f = -JDJf = -J(i\phi_k(D)Jf) = i\phi_k(D)f$, and hence if $\mu \in E_k \cap F_j$ we have

$$D\mu = i\phi_k(D)\mu, \qquad D_R\mu = i\phi_j(D)\mu. \tag{4.5}$$

Let f_{k1} $(1 \leqq k \leqq d_E)$ be a unit vector in $E_1 \cap F_k$. Then $(f_{k1})^-$ $(1 \leqq k \leqq d_E)$ is an orthonormal basis for $\bar{E}_1$. The matrix coordinates of the left regular representation L restricted to $\bar{E}_1$ relative to the basis $(f_{k1})^-$ are given by $\mu_{kj}(a) = (L(a)\bar{f}_{j1}, \bar{f}_{k1}) = f_{k1} * f_{j1}^*(a)$ so $\mu_{kj} \in E_j \cap F_k$. It follows from (4.5) that the functions μ_{kj} have the properties stated in the lemma.

For the rest of the paper $G = SU(2)$ and $\mathfrak{G}$ is the Lie algebra of G. Let Q be the projection onto the space of class functions in $L^2(G)$. Then for any continuous function g on G, Qg is the continuous class function on G defined by

$$Qg(a) = \int_G g(bab^{-1})\, db.$$

If μ_{ij} is a coordinate function of an irreducible representation of G with character χ_n, then using the relations in [5, p. 73] we can show that

$$Q\mu_{ij} = \delta_{ij}\chi_n/n. \tag{4.6}$$

For any $D \in \mathfrak{G}$ and any $T \in [0, \pi]$ the map μ_{TD} of $C(G) \to C$ defined by

$$\mu_{TD}: g \to (Q(gD\chi_2))(x_T), \tag{4.7}$$

where $x_T = \operatorname{diag}(e^{iT}, e^{-iT})$ is easily seen to be a measure on G.

Proposition 4.8. Let $t \in [-2, 2]$, let λ_t be the function on G defined by (3.31), and let $D \in \mathfrak{G}$. Then

$$D\lambda_t = (1/\pi)(\sin T)\mu_{TD}, \qquad T = \arccos(\tfrac{1}{2}t) \tag{4.9}$$

where μ_{TD} is defined in (4.7), i.e.,

$$\pi(Dg, \lambda_t) = -(\sin T)(Q(gD\chi_2))(x_T) \tag{4.10}$$

for all $g \in C^\infty(G)$.

Proof. The result is clear if $t = \pm 2$, so assume $t \in (-2, 2)$. Let $\mathfrak{A}$ be the abelian subalgebra of $\mathfrak{G}$ generated by D, and for each positive integer n let $a \to (\mu_{kj}^n(a))$ be an irreducible n-dimensional matrix representation of G and let $\phi_1^n, \ldots, \phi_n^n$ be real linear functionals on $\mathfrak{A}$ having the properties described in (4.2)–(4.4). To prove (4.10) it is sufficient to show that

$$\pi i\phi_j^n(D)(\mu_{kj}^n, \lambda_t) = -(\sin T)(Q(\mu_{kj}^n D\chi_2))(x_T) \tag{4.11}$$

for $1 \leqq n < \infty$ and $1 \leqq k, j \leqq n$. For $n > 1$ we obtain from (4.6), (3.28), and (3.32)

$$(\mu_{kj}^n, \lambda_t) = (\mu_{kj}^n, Q\lambda_t) = (Q\mu_{kj}^n, \lambda_t) = \delta_{kj}(\chi_n, \lambda_t)/n$$
$$= \delta_{kj}((\sin (n-1)T/\pi n(n-1)) - (\sin (n+1)T/\pi n(n+1))),$$

and hence for all $n > 1$

$$\pi n i\phi_j^n(D)(\mu_{kj}^n, \lambda_t) = i\delta_{kj}\phi_j^n(D) \sin T((\chi_{n-1}(x_T)/(n-1)) - (\chi_{n+1}(x_T)/(n+1))). \tag{4.12}$$

This equation also holds for $n=1$ if we agree to set $\chi_{n-1}/(n-1)=0$ when $n=1$. Hence we will have (4.11) if we show that

$$nQ(\mu_{kj}^n D\chi_2) = i\delta_{kj}\phi_j^n(D)(\chi_{n+1}/(n+1) - \chi_{n-1}/(n-1)). \tag{4.13}$$

We prove (4.13) by showing that both sides of the equation have the same inner product with μ_{sr}^m for $1 \leqq m < \infty$ and $1 \leqq s, r \leqq m$. By the usual orthogonality relations for coordinate functions we get

$$\begin{aligned} &(i\delta_{kj}\phi_j^n(D)(\chi_{n+1}/(n+1) - \chi_{n-1}/(n-1)), \mu_{sr}^m) \\ &\qquad = i\delta_{kj}\, \delta_{sr}\phi_j^n(D)(\delta_{n+1,m} - \delta_{n-1,m})/m^2. \end{aligned} \tag{4.14}$$

Now use (3.4) to get

$$\begin{aligned} (nQ(\mu_{kj}^n D\chi_2), \mu_{sr}^m) &= n(\mu_{kj}^n D\chi_2, Q\mu_{sr}^m) = n\delta_{sr}(\mu_{kj}^n, D\chi_2 \cdot \chi_m)/m \\ &= n\delta_{sr}(\mu_{kj}^n, D(\chi_{m+1} - \chi_{m-1}))/m^2 \\ &= n\delta_{sr}(D\mu_{kj}^n, \chi_{m-1} - \chi_{m+1})/m^2 \\ &= n\delta_{sr}(i\phi_j^n(D)\mu_{kj}^n, \chi_{m-1} - \chi_{m+1})/m^2 \\ &= i\delta_{sr}\delta_{kj}\phi_j^n(D)(\delta_{n,m-1} - \delta_{n,m+1})/m^2. \end{aligned} \tag{4.15}$$

Compare (4.14) and (4.15) and see that we have proved (4.13) which completes the proof of Proposition 4.8.

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